FURTHER MATHS

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In this booklet are topics and skills that are an introduction to some of the areas studied in AS Further Mathematics. For each week, there are hyperlinks to videos from Advanced Maths Support Programme (AMSP) that can be used for learning or revision to assist you in the completion of the exercises. Before the exercises, there is an introduction to the topic to give you an insight into its uses and application in Mathematics. At the end of the pack there are worked solutions for each section so you can check your progress with topics.

The growth of the number system

In your learning of mathematics, you have come across different types of number at different stages. Each time you were introduced to a new set of numbers, this allowed you to solve a wider range of problems.

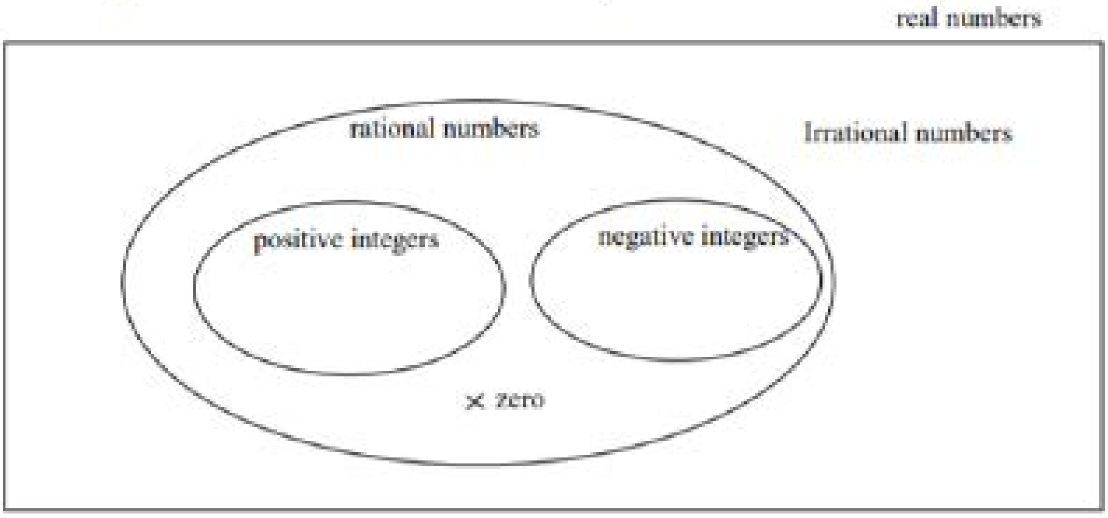
The first numbers that you came across were the counting numbers (natural numbers). These allowed you to solve equations like $x + 2 = 5$.

Later you would meet negative numbers, which allowed you to solve equations like $x + 5 = 2$, and rational numbers, which meant you could solve equations like $2x = 5$.

When irrational numbers were included, you could solve equations like $x^2 = 2$.

However, there are still equations which you cannot solve, such as $x^2 = -4$. You know that there are no real numbers which satisfy this equation. However, this equation, and others like it, can be solved using imaginary numbers, which are based on the number i, which is defined as $\sqrt{-1}$.

The diagram below shows the relationships between different types of number.



This diagram deals with the real numbers, which include all numbers which you have come across until now. Notice that the positive and negative integers (whole numbers) are subsets of the rational numbers. This means that all integers are also rational numbers, but there are other rational numbers which are not integers, such as $\frac{1}{2}$ or $-\frac{7}{11}$. Similarly, all rational numbers are real numbers, but there are other real numbers which are not rational, such as $\sqrt{3}$ and π.

In this topic you will see that the real numbers are also a subset of a larger set called the complex numbers. You will be looking at numbers which lie outside the set of real numbers. Complex numbers involve both real and imaginary numbers.

Week 1 continued

Tutorial videos
Intro to complex numbers - https://youtu.be/e1XsCe0Sfr0
Adding and subtracting complex numbers - https://youtu.be/-74cX8rLqto
Multiplying complex numbers - https://youtu.be/LSUO5ck4aP4
Dividing complex numbers - https://youtu.be/mL5AKNQ94ko
Real, imaginary and conjugate definitions - https://youtu.be/A_rEiF8TUO0

Introduction to Complex Numbers Exercise

1. Find the roots of the following equations:
   (i) $z^2 + 25 = 0$ (ii) $4z^2 + 9 = 0$
   (iii) $z^2 - 2z + 2 = 0$ (iv) $4z^2 + 4z + 5 = 0$

2. Two complex numbers $4 - 3\mathrm{i}$ and $2 + \mathrm{i}$ are denoted by z and w respectively. Find, giving your answers in the form $x + y\mathrm{i}$.
   (i) $2z - 3w$
   (ii) zw
   (iii) $(\mathrm{i}z)^2$
   (iv) z^*w

3. In each of the following cases find
   (a) $z_1 + z_2$ (b) $z_1 - z_2$ (c) z_1z_2 (d) z_1^*
   (e) z_2^* (f) $z_1^* + z_2^*$ (g) $z_1^* - z_2^*$ (h) $z_1^*z_2^*$

   (i) $z_1 = 2 + 3\mathrm{i}$; $z_2 = 1 - 2\mathrm{i}$
   (ii) $z_1 = -2\mathrm{i}$; $z_2 = 3 + \mathrm{i}$
   What do you notice about the results?

4. Find the quadratic equation which has roots $2 + 3\mathrm{i}$ and $2 - 3\mathrm{i}$.

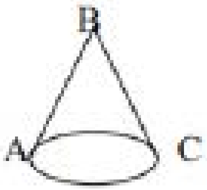
5. Express these complex numbers in the form $x + y\mathrm{i}$.
   (a) $\dfrac{2}{3+\mathrm{i}}$ (b) $\dfrac{2-\mathrm{i}}{1+2\mathrm{i}}$

6. Solve the equation $(2+\mathrm{i})z = 3 + 4\mathrm{i}$.

Week 2 – Introduction to Matrices

Matrices

A matrix is simply a way of storing information. For example, the diagram below shows a map of the roads linking three towns A, B and C. The corresponding '*direct route*' matrix is shown beside it.



$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \quad A \quad B \quad C \\ \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \end{array}$$

In this section you learn to add and subtract matrices, to multiply a matrix by a number and to multiply two matrices.

Matrices are classified by number of rows and the number of columns they have. The matrix above has 3 rows and 3 columns, it is a 3×3 matrix (read as '3 by 3').

A matrix with m rows and n columns is an $m \times n$ matrix. This is called the order of the matrix.

A **square matrix** is a matrix with the same number of rows as columns.

You can add or subtract matrices provided they have the **same order**.

Tutorial videos
Basic Matrix operations - https://youtu.be/CibmDO2sYn0
Multiply Matrices - https://youtu.be/eWIRzlm59XE
Properties of Matrix multiplications - https://youtu.be/ba_aDWCiFBA
Zero and identity matrices - https://youtu.be/6JfM0ruhJVs

Multiplying matrices

Multiplying matrices is an important skill which you must master. It takes a bit of getting used to, but after plenty of practice you will find it quite straightforward.

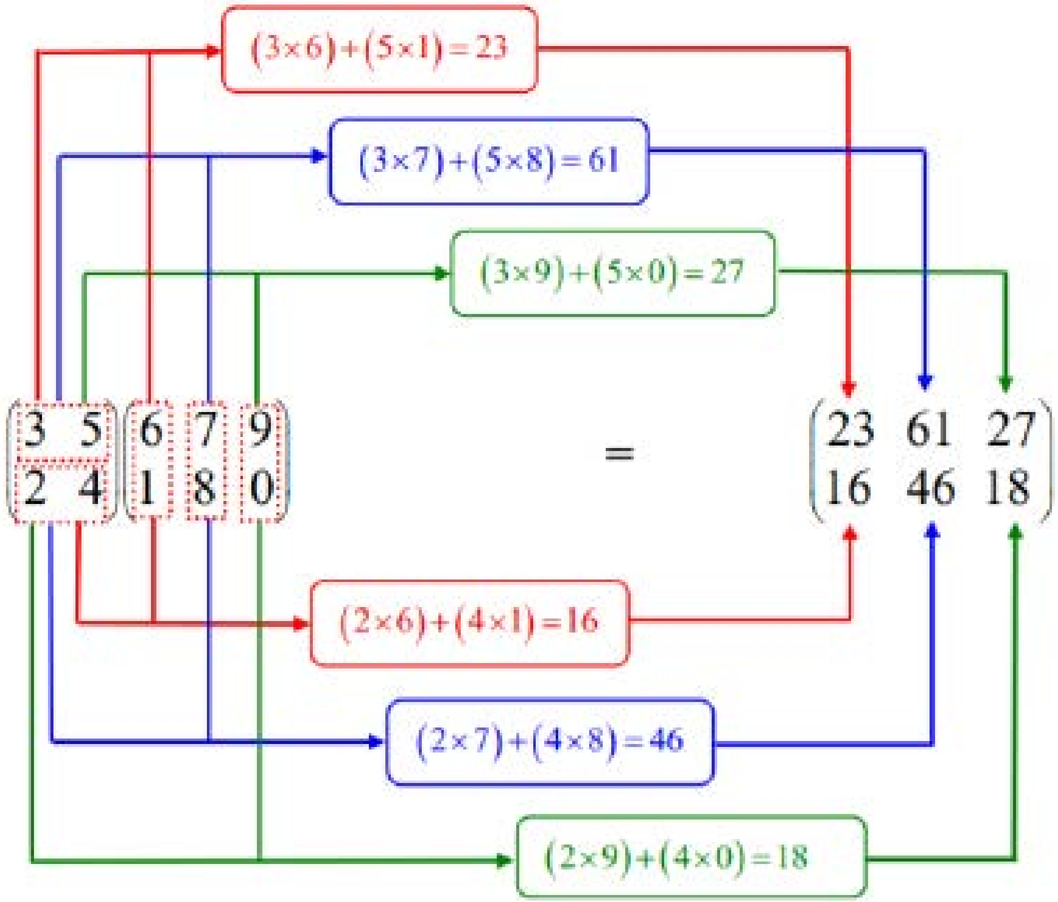
The important points to remember are:

- Use each row of the first matrix with each column of the second.
- When you are using row a of the first matrix with column b of the second matrix, the result gives you the element in row a, column b of the product matrix.
- To multiply matrices, the number of columns in the first matrix must be the same as the number of rows in the second matrix. If this is not the case, the matrices do not conform and cannot be multiplied.

The diagram below shows the steps used when carrying out the multiplication

You use each row of the first (i.e. left) matrix with each column, in turn, of the second matrix.

A similar technique applies to all matrix multiplications.



Properties of matrix multiplication

Make sure that you know the important properties of matrix multiplication:

- Matrices must be conformable for multiplication
- Matrix multiplication is not commutative
- Matrix multiplication is associative
- Matrix multiplication is distributive

The identity matrix

The matrix **I** = $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ is called the 2×2 **identity matrix** because when you multiply any 2×2 matrix **A** by **I** you get **A** as the answer.

I acts like the number 1 in the multiplication of numbers.

This means that for any 2×2 matrix **A**:

$$\mathbf{IA} = \mathbf{AI} = \mathbf{A}.$$

1. $\mathbf{A}=\begin{pmatrix}2 & -3\\ -1 & 5\end{pmatrix}$ $\mathbf{B}=\begin{pmatrix}-3 & -1\\ 2 & 7\end{pmatrix}$ $\mathbf{C}=\begin{pmatrix}2 & 3 & -4\\ -1 & 2 & 5\end{pmatrix}$ $\mathbf{D}=\begin{pmatrix}-1 & -4 & 2\\ -3 & 5 & 6\end{pmatrix}$

   Calculate, if possible,
   (i) $\mathbf{A}+2\mathbf{B}$ (ii) $\mathbf{C}-\mathbf{D}$ (iii) $3\mathbf{A}-2\mathbf{C}$ (iv) $3\mathbf{D}-\mathbf{C}$

2. $\mathbf{A}=\begin{pmatrix}2 & 1\\ -3 & 4\end{pmatrix}$ $\mathbf{B}=\begin{pmatrix}-1 & 3 & 2\\ 5 & 1 & -2\end{pmatrix}$ $\mathbf{C}=\begin{pmatrix}3 & -1\\ 1 & 2\end{pmatrix}$ $\mathbf{D}=\begin{pmatrix}4 & -1\\ 2 & 5\\ -3 & 1\end{pmatrix}$

   Calculate, if possible, the following
   (i) **AB** (ii) **AC** (iii) **BC** (iv) **BD**

3. The matrices **A** and **B** are defined by

   $\mathbf{A}=\begin{pmatrix}2 & 3\\ 3 & 2\end{pmatrix}$ $\mathbf{B}=\begin{pmatrix}0 & 3\\ 3 & 0\end{pmatrix}$

   (i) Calculate
   (a) $\mathbf{A}+\mathbf{B}$ (b) **AB**

   (ii) Show that $\mathbf{A}+\mathbf{B}-\mathbf{AB}=m\mathbf{I}$, where m is an integer and **I** is the 2 x 2 identity matrix.

4. The matrices **A**, **B** and **C** are given by $\mathbf{A}=\begin{pmatrix}1 & 4\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}-3\\ 2\end{pmatrix}$ and $\mathbf{C}=\begin{pmatrix}2 & -1\end{pmatrix}$

   Find (i) $2\mathbf{A}+\mathbf{C}$ (ii) **AB** (iii) **BC**

5. If $\mathbf{A}=\begin{pmatrix}3 & 1\\ x & 2\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}6 & 2\\ 4 & y\end{pmatrix}$ find the values of x and y given that $\mathbf{AB}=\mathbf{BA}$.

6. $\mathbf{M}=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$ and $\mathbf{N}=\begin{pmatrix}2 & -2\\ -1 & 3\end{pmatrix}$.

   Find $\mathbf{M}^2-\mathbf{N}^2$ and $(\mathbf{M}+\mathbf{N})(\mathbf{M}-\mathbf{N})$ and explain why your results are not equal.

Week 3 – Sorting Algorithms

Algorithms

An algorithm is simply a set of precise instructions. Although in this chapter (and throughout Decision Maths) you will be working through algorithms by hand, in real life algorithms are usually programmed into computers.

Sorting algorithms

There are a number of different algorithms that can be used to sort a list into order. You will be learning the bubble sort and the quick sort algorithms.

The Sorting spreadsheet available on Integral is useful for investigating these algorithms.

To make efficient use of the spreadsheet, print these notes out and experiment with the spreadsheet whilst referring to them.The spreadsheet will really help you to understand how the sorting algorithms work.Download the spreadsheet from Integral and save it. This will enable you to spend time experimenting with it off-line.

When you open the spreadsheet, a dialogue box will appear asking if you want to enable macros. You must select 'Enable Macros' or the spreadsheet will not function.

The spreadsheet contains three worksheets: Bubble, Quick, Data.

The first two worksheets demonstrate the two sorting algorithms. They can sort lists up to ten items long.

The Data worksheet contains data sets which you may sort using the other worksheets. To do this, cut and paste the data set you require into the first column of any of the sorting worksheets, replacing the list that is already there. You may also enter your own lists into the sorting worksheets.

Tutorial videos
Bubble sort - https://youtu.be/pskgS27Ctr4
Quick sort - https://youtu.be/xhZuhxmE4q8

The bubble sort

To sort in ascending order, on the first pass, the top item in the list is compared with the second. If it is larger, the two items are exchanged. If it is smaller, the items remain where they are. The second item in the list is then compared to the third. As before, if the second is larger than the third, the items are exchanged, if it is smaller, the items remain where they are. This procedure is followed down through the list until the largest item has 'bubbled down' to the top of the list. This is the end of the first pass. On the worksheet items that are being compared are highlighted in pink. The item that arrives at the bottom after the first pass is highlighted in turquoise, to indicate that it is now in the correct position.

The second pass is the same as the first pass, except the last item in the list is not considered, since it must be in the right position after the first pass. Subsequent passes all work in the same way, but the number of items being considered goes down by one each time as the sorted portion of the list (highlighted in turquoise) grows. The process continues until either there have been one fewer passes than the number of items in the list, by which time the list must be sorted, or a pass takes place which involves no swaps, which means the items must already be in the correct order.

The quick sort

At first this algorithm can seem difficult to understand, but it is also, often, the most efficient.

The number which is chosen for the pivot in the quick sort algorithm can be selected in many different ways. In the textbook, the middle number in the list is chosen, and this is the method you will be expected to use in your examination. However, on the spreadsheet the first number in the list is chosen. Other than that, the spreadsheet works in the same way.

On the worksheet, the first number in the list or sublist under consideration is highlighted in pink and is compared in turn with the other numbers in the list or sublist. Those smaller than it are added to the end of a new sublist above it. Those larger than it are placed at the end of a new sublist below it. Once the whole list or sublist has been worked through the number that was highlighted pink it highlighted turquoise, to indicate it is now in the correct position in the overall list.
The quick sort only uses comparisons to find the correct position for the item at the top of the sub-list under consideration. No exchanges are made.

It is a good idea to run the worksheet through several different initial lists, to see how it behaves.

1. Use a bubble sort to put these numbers into descending order:

   56 78 43 34 86 23

   Write down
   - (i) The resulting list at the end of each pass
   - (ii) The total number of comparisons made
   - (iii) The total number of swaps made

2. Use a bubble sort to arrange these letters into alphabetical order

   Y H Q P G I K

   showing the resulting list after each pass.

3. Use a quick sort to arrange these numbers into *descending* order

   67 78 45 89 64 34 90 77

   Show enough working to make it clear that you have applied the algorithm correctly.

4. (i) The following list of numbers is to be sorted into descending order.

   35 23 10 46 24 11

   Use the Bubble sort algorithm to obtain a sorted list, giving the state of the list at the end of each pass.

5. A machinist has to cut the following seven lengths (in centimeters) of steel tubing.

   150 104 200 60 184 84 120

   (i) Perform a quick sort to put the seven lengths in descending order.

Week 4 – Algorithms and flow charts

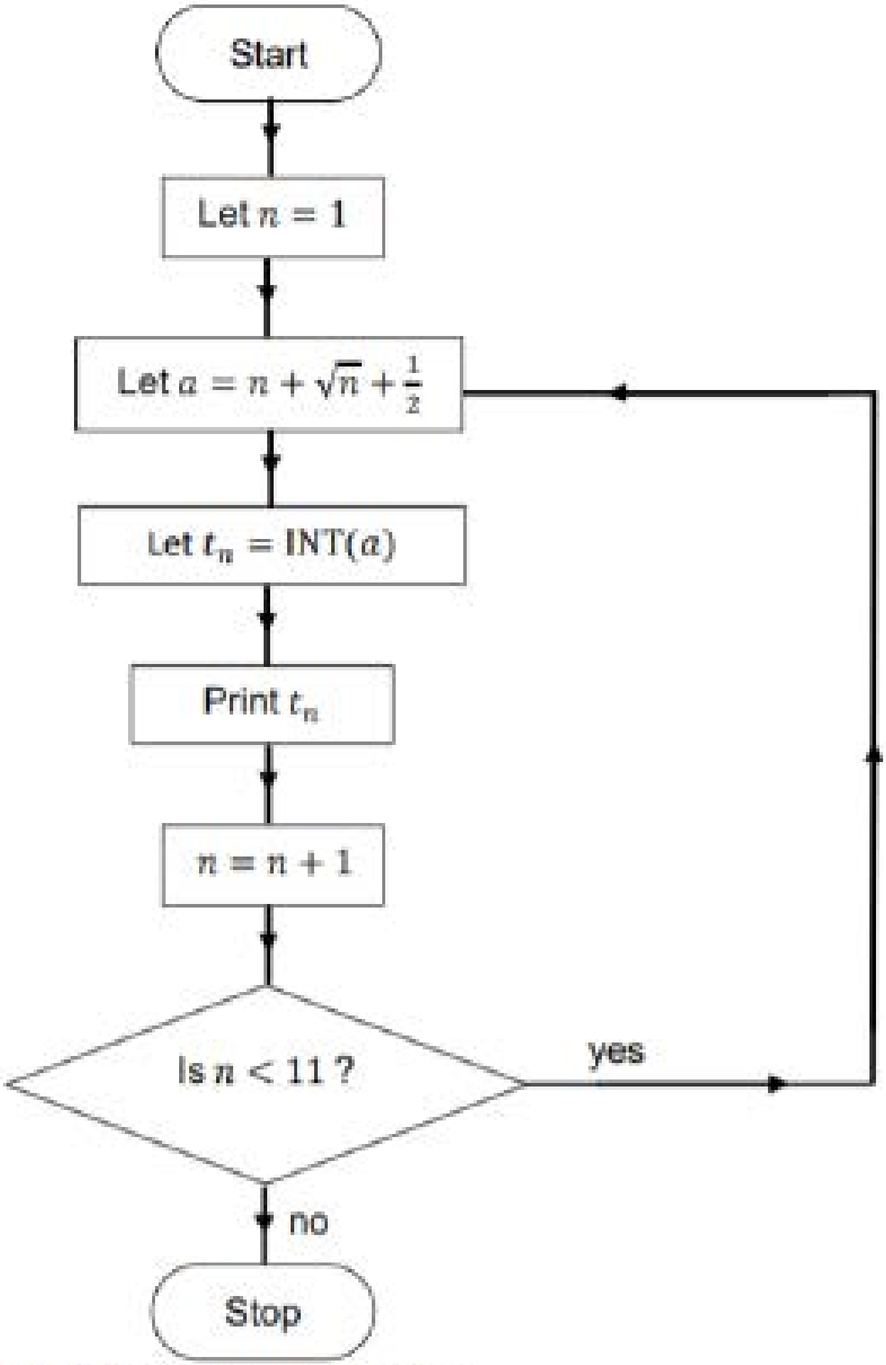
Example of a flow chart used in the Big Bang Theory:
https://www.youtube.com/watch?v=jWWOM53Zh20

Tutorial video

Algorithms and flow charts - https://youtu.be/NzfhV2Zm-Dw

Algorithms and flow charts Exercise

1. Apply the algorithm shown in the flowchart.
   Note: Int(a) is the integer part of a
   e.g. if $a = 2.78$ then INT(a) = 2, if $a = 17.04$ then INT(a) = 17.



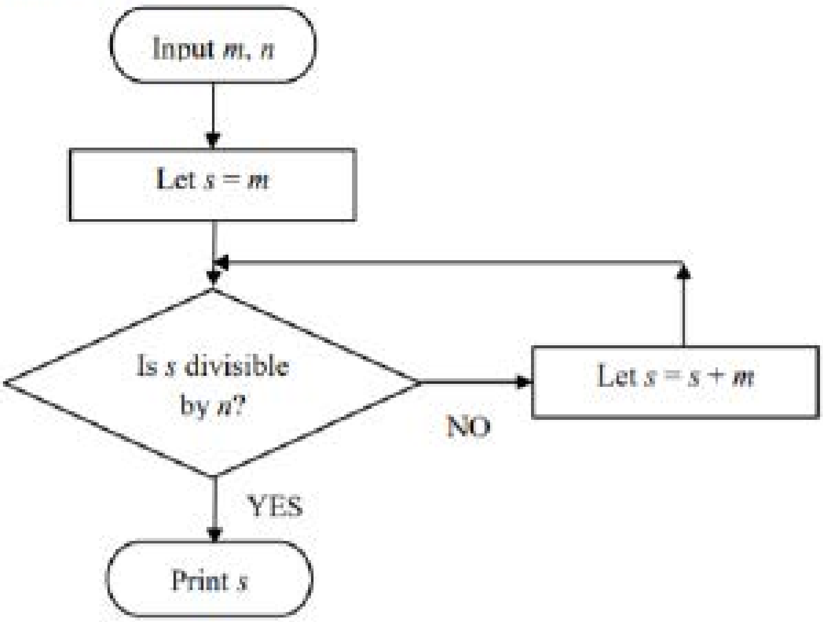
Write down the output at each stage.
What do you notice about the sequence of numbers?

2. Work through the algorithm below with input
   (i) 5
   (ii) 2001

Step 1 Input a
Step 2 $b = a + 4$
Step 3 $b = a \times b$
Step 4 $b = b + 4$
Step 5 $b = \sqrt{b}$
Step 6 $b = b + 5$
Step 7 $b = b - a$
Step 8 If $b = 7$ go to Step 11
Step 9 Write 'Help'
Step 10 End
Step 11 Write 'Maths is magic'
Step 12 End

3. For the algorithm in question 2:
   (i) Explain why there are two Steps with an End command.
   (ii) Find an input where the algorithm fails. Adapt the algorithm to cope with this.

4. The following flow chart defines an algorithm which operates on two positive integers, m and n.



   (i) Experiment with pairs of small positive integers. What does the algorithm achieve?
   (ii) Does it matter in which order m and n are inputted?
   (iii) Explain why the algorithm will always stop.

Solutions

Week 1 – Introduction to complex number solutions

1\. (i) $z^2 + 25 = 0$

$z^2 = -25$

$z = \pm 5i$

(ii) $4z^2 + 9 = 0$

$z^2 = -\frac{9}{4}$

$z = \pm\frac{3}{2}i$

(iii) $z^2 - 2z + 2 = 0$

$z = \frac{2 \pm \sqrt{4 - 4 \times 1 \times 2}}{2}$

$= \frac{2 \pm \sqrt{-4}}{2}$

$= \frac{2 \pm 2i}{2}$

$= 1 \pm i$

(iv) $4z^2 + 4z + 5 = 0$

$z = \frac{-4 \pm \sqrt{16 - 4 \times 4 \times 5}}{8}$

$= \frac{-4 \pm \sqrt{-64}}{8}$

$= \frac{-4 \pm 8i}{8}$

$= -\frac{1}{2} \pm i$

2\. (i) $2z - 3w = 2(4 - 3i) - 3(2 + i)$

$= 8 - 6i - 6 - 3i$

$= 2 - 9i$

(ii) $zw = (4 - 3i)(2 + i)$

$= 8 - 6i + 4i - 3i^2$

$= 8 - 2i + 3$

$= 11 - 2i$

(iii) $iz = i(4 - 3i) = 4i - 3i^2 = 4i + 3$

$(iz)^2 = (4i + 3)^2$

$= 16i^2 + 24i + 9$

$= -16 + 24i + 9$

$= -7 + 24i$

(iv) $z^*w = (4 - 3i)^*(2 + i)$

$= (4 + 3i)(2 + i)$

$= 8 + 6i + 4i + 3i^2$

$= 8 + 10i - 3$

$= 5 + 10i$

Week 1 continued

3. (i) (a) $z_1 + z_2 = 2 + 3i + 1 - 2i = 3 + i$

(b) $z_1 - z_2 = 2 + 3i - 1 + 2i = 1 + 5i$

(c) $z_1z_2 = (2 + 3i)(1 - 2i) = 2 - 4i + 3i + 6 = 8 - i$

(d) $z_1^* = 2 - 3i$

(e) $z_2^* = 1 + 2i$

(f) $z_1^* + z_2^* = 2 - 3i + 1 + 2i = 3 - i$

(g) $z_1^* - z_2^* = 2 - 3i - 1 - 2i = 1 - 5i$

(h) $z_1^* z_2^* = (2 - 3i)(1 + 2i) = 2 + 4i - 3i + 6 = 8 + i$

(ii) (a) $z_1 + z_2 = -2i + 3 + i = 3 - i$

(b) $z_1 - z_2 = -2i - 3 - i = -3 - 3i$

(c) $z_1z_2 = -2i(3 + i) = -6i + 2 = 2 - 6i$

(d) $z_1^* = 2i$

(e) $z_2^* = 3 - i$

(f) $z_1^* + z_2^* = 2i + 3 - i = 3 + i$

(g) $z_1^* - z_2^* = 2i - (3 - i) = -3 + 3i$

(h) $z_1^* z_2^* = 2i(3 - i) = 6i + 2 = 2 + 6i$

$z_1^* + z_2^* = (z_1 + z_2)^*$

$z_1^* - z_2^* = (z_1 - z_2)^*$

$z_1^* z_2^* = (z_1z_2)^*$

4. $z = 2 \pm 3i$

$(z - 2) = \pm 3i$

$(z - 2)^2 = -9$

$z^2 - 4z + 4 = -9$

$z^2 - 4z + 13 = 0$

5. (i) $\frac{2}{3+i} = \frac{2(3-i)}{(3+i)(3-i)}$

$= \frac{2(3-i)}{9+1}$

$= \frac{2(3-i)}{10} = \frac{3-i}{5}$

(ii) $\frac{2-i}{1+2i} = \frac{(2-i)(1-2i)}{(1+2i)(1-2i)}$

$= \frac{2-i-4i-2}{1+4}$

$= \frac{-5i}{5} = -i$

6. $(2+i)z = 3+4i$

$z = \frac{3+4i}{(2+i)}$

$= \frac{(3+4i)(2-i)}{(2+i)(2-i)}$

$= \frac{6-3i+8i+4}{4+1}$

$= \frac{10+5i}{5} = 2+i$

7. (i) The other root is $1 - 2i$

(ii) $z = 1 \pm 2i$

$(z - 1) = \pm 2i$

$(z - 1)^2 = -4$

$z^2 - 2z + 1 = -4$

$z^2 - 2z + 5 = 0$

So $a = -2, b = 5$

Week 2 – Introduction to Matrices solutions

1. (i) $A+2B=\begin{pmatrix}2 & -3\\ -1 & 5\end{pmatrix}+2\begin{pmatrix}-3 & -1\\ 2 & 7\end{pmatrix}=\begin{pmatrix}2 & -3\\ -1 & 5\end{pmatrix}+\begin{pmatrix}-6 & -2\\ 4 & 14\end{pmatrix}$

$=\begin{pmatrix}-4 & -5\\ 3 & 19\end{pmatrix}$

(ii) $C-D=\begin{pmatrix}2 & 3 & -4\\ -1 & 2 & 5\end{pmatrix}-\begin{pmatrix}-1 & -4 & 2\\ -3 & 5 & 6\end{pmatrix}$

$=\begin{pmatrix}3 & 7 & -6\\ 2 & -3 & -1\end{pmatrix}$

(iii) cannot be done as A and C do not have the same order

(iv) $3D-C=3\begin{pmatrix}-1 & -4 & 2\\ -3 & 5 & 6\end{pmatrix}-\begin{pmatrix}2 & 3 & -4\\ -1 & 2 & 5\end{pmatrix}$

$=\begin{pmatrix}-3 & -12 & 6\\ -9 & 15 & 18\end{pmatrix}-\begin{pmatrix}2 & 3 & -4\\ -1 & 2 & 5\end{pmatrix}$

$=\begin{pmatrix}-5 & -15 & 10\\ -8 & 13 & 13\end{pmatrix}$

2. (i) $AB=\begin{pmatrix}2 & 1\\ -3 & 4\end{pmatrix}\begin{pmatrix}-1 & 3 & 2\\ 5 & 1 & -2\end{pmatrix}=\begin{pmatrix}3 & 7 & 2\\ 23 & -5 & -14\end{pmatrix}$

(ii) $AC=\begin{pmatrix}2 & 1\\ -3 & 4\end{pmatrix}\begin{pmatrix}3 & -1\\ 1 & 2\end{pmatrix}=\begin{pmatrix}7 & 0\\ -5 & 11\end{pmatrix}$

(iii) BC cannot be calculated as the matrices are not conformable (the number of columns in B is not the same as the number of rows in C)

(iv) $BD=\begin{pmatrix}-1 & 3 & 2\\ 5 & 1 & -2\end{pmatrix}\begin{pmatrix}4 & -1\\ 2 & 5\\ -3 & 1\end{pmatrix}=\begin{pmatrix}-4 & 18\\ 28 & -2\end{pmatrix}$

3. (i) $A+B=\begin{pmatrix}2 & 3\\ 3 & 2\end{pmatrix}+\begin{pmatrix}0 & 3\\ 3 & 0\end{pmatrix}=\begin{pmatrix}2 & 6\\ 6 & 2\end{pmatrix}$

$AB=\begin{pmatrix}2 & 3\\ 3 & 2\end{pmatrix}\begin{pmatrix}0 & 3\\ 3 & 0\end{pmatrix}=\begin{pmatrix}9 & 6\\ 6 & 9\end{pmatrix}$

(ii) $A+B-AB=\begin{pmatrix}2 & 6\\ 6 & 2\end{pmatrix}-\begin{pmatrix}9 & 6\\ 6 & 9\end{pmatrix}=\begin{pmatrix}-7 & 0\\ 0 & -7\end{pmatrix}=-7\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=-7I$

4. (i) $2A + C = 2\begin{pmatrix} 1 & 4 \end{pmatrix} + \begin{pmatrix} 2 & -1 \end{pmatrix}$

$= \begin{pmatrix} 2 & 8 \end{pmatrix} + \begin{pmatrix} 2 & -1 \end{pmatrix}$

$= \begin{pmatrix} 4 & 7 \end{pmatrix}$

(ii) $AB = \begin{pmatrix} 1 & 4 \end{pmatrix}\begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \end{pmatrix}$

(iii) $BC = \begin{pmatrix} -3 \\ 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \end{pmatrix} = \begin{pmatrix} -6 & 3 \\ 4 & -2 \end{pmatrix}$

5. $AB = \begin{pmatrix} 3 & 1 \\ x & 2 \end{pmatrix}\begin{pmatrix} 6 & 2 \\ 4 & y \end{pmatrix} = \begin{pmatrix} 22 & 6+y \\ 6x+8 & 2x+2y \end{pmatrix}$

$BA = \begin{pmatrix} 6 & 2 \\ 4 & y \end{pmatrix}\begin{pmatrix} 3 & 1 \\ x & 2 \end{pmatrix} = \begin{pmatrix} 18+2x & 10 \\ 12+xy & 4+2y \end{pmatrix}$

$AB = BA \Rightarrow \begin{pmatrix} 22 & 6+y \\ 6x+8 & 2x+2y \end{pmatrix} = \begin{pmatrix} 18+2x & 10 \\ 12+xy & 4+2y \end{pmatrix}$

$22 = 18 + 2x \quad \Rightarrow x = 2$

$6 + y = 10 \quad \Rightarrow y = 4$

Check: $6x + 8 = 12 + 8 = 20 \quad 12 + xy = 12 + 8 = 20$

$2x + 2y = 4 + 8 = 12 \quad 4 + 2y = 4 + 8 = 12$

6. $M^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$

$N^2 = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 6 & -10 \\ -5 & 11 \end{pmatrix}$

$M^2 - N^2 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 6 & -10 \\ -5 & 11 \end{pmatrix} = \begin{pmatrix} -5 & 14 \\ 5 & -10 \end{pmatrix}$

$M + N = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ -1 & 4 \end{pmatrix}$

$M - N = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 1 & -2 \end{pmatrix}$

$(M+N)(M-N) = \begin{pmatrix} 3 & 0 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} -1 & 4 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} -3 & 12 \\ 5 & -12 \end{pmatrix}$

$(M+N)(M-N) = M^2 + NM - MN - N^2$

Since matrix multiplication is not commutative, $NM \neq MN$.

Week 3 – Sorting Algorithms solutions

1. (i)

Pass 1:	78	56	43	86	34	23	
Pass 2:	78	56	86	43	34	23	
Pass 3:	78	86	56	43	34	23	
Pass 4:		86	78	56	43	34	23
Pass 5:	86	78	56	43	34	23	

Although "nothing happens" in pass 5 as far as reordering the list is concerned, it must be there as the terminating condition of the algorithm is that there are no swaps in a pass.
As swaps occurred in Pass 4, it could not be the final pass.
Something actually did happen in the pass, one comparison was made.

(ii) Total number of comparisons made $= 5 + 4 + 3 + 2 + 1 = 15$

(iii) Total number of swaps made $= 2 + 1 + 1 + 1 = 5$

2.

Pass 1:		H	Q	P	G	I	K	Y
Pass 2:	H	P	G	I	K	Q	Y	
Pass 3:	H	G	I	K	P	Q	Y	
Pass 4:		G	H	I	K	P	Q	Y
Pass 5:	G	H	I	K	P	Q	Y	

3. $\frac{8+1}{2} = 4.5$ use position 5

67	78	45	89	64	34	90	77
67	78	89	90	77	64	45	34
90	89	67	78	77	64	45	34
90	89	78	67	77	64	45	34
90	89	78	77	67	64	45	34

The algorithm terminates when every term has been chosen as a pivot. This has now happened (and the list is in order).

Week 3 continued

4. (i)

Pass 1:	35	23	46	24	11	10
Pass 2:	35	46	24	23	11	10
Pass 3:	46	35	24	23	11	10
Pass 4:	46	35	24	23	11	10

5. (i)

150	104	200	60	184	84	120
150	104	200	184	84	120	60
200	184	150	104	84	120	60
200	184	150	104	120	84	60
200	184	150	120	104	84	60
200	184	150	120	104	84	60
200	184	150	120	104	84	60

Week 4 – Algorithms and flow charts solutions

1. Here is a trace of the algorithm showing the values at each stage

n	a	t_n	Output	$n = n + 1$	$n < 11$
1	2.5	2	$t_1 = 2$	2	yes
2	3.914214	3	$t_2 = 3$	3	yes
3	5.232051	5	$t_3 = 5$	4	yes
4	6.5	6	$t_4 = 6$	5	yes
5	7.736068	7	$t_5 = 7$	6	yes
6	8.949490	8	$t_6 = 8$	7	yes
7	10.145751	10	$t_7 = 10$	8	yes
8	11.328427	11	$t_8 = 11$	9	yes
9	12.5	12	$t_9 = 12$	10	yes
10	13.662278	13	$t_{10} = 13$	11	no

It produces all of the positive integers except the perfect square numbers.

2. (i) Maths is magic
(ii) Maths is magic

3. (i) At step 8 the algorithm divides in two, so there are two endpoints
(ii) $a = -10$
The algorithm could be adapted by replacing a with abs(a) between steps 1 and 2.
(other adaptations are possible)

4. (i) It prints the lowest common multiple of m and n
(ii) No. In both cases the algorithm stops at the LCM (lowest common multiple) of m and n
(iii) The algorithm is checking whether successive multiples of m are divisible by n. Certainly mn is divisible by n, so the algorithm will stop then if not before.